Computing control Lyapunov functions with neural networks

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based on joint work with Mario Sperl (Bayreuth) and Debasish Chatterjee (Mumbai)

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InterCoML
Control Theory & Machine Learning

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Setting

We consider nonlinear control systems in continuous time

$$\dot{x}(t) := \frac{d}{dt}x(t) = f(x(t), u(t)), \quad x(0) = x_0,$$

or in discrete time

$$x^+(t) := x(t+1) = g(x(t), u(t)), \quad x(0) = x_0,$$

where $f, g \colon \mathbb{R}^n \times U \to \mathbb{R}^n$ is a controlled vector field or map, respectively

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The feedback allows to react to deviations of the real world system from its mathematical model used for designing the control (of course, F must be designed properly such that the control reacts reasonably to such deviations)

Lyapunov functions

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We first explain Lyapunov functions without control and in continuous time

We consider autonomous ordinary differential equations (ODEs)

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$$\begin{aligned} \alpha_1(\|x\|) \ \leq \ V(x) \ &\leq \ \alpha_2(\|x\|) \\ DV(x)f(x) \ &\leq \ -\alpha_3(\|x\|) \end{aligned}$$

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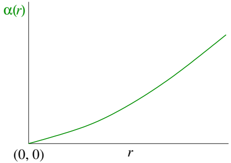
$\alpha \in \mathcal{K}_\infty$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$, unbounded



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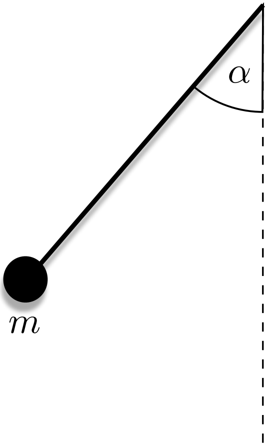
Example: Mathematical Pendulum



$x_1 = \alpha =$ angle

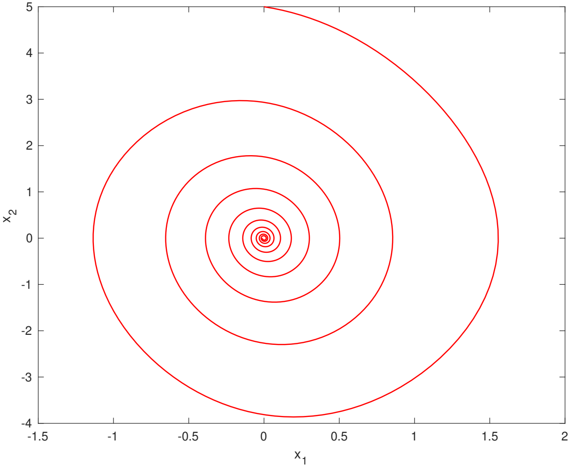
$x_2 =$ angular velocity

$\rightsquigarrow$ ordinary differential equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g\sin(x_1) - \frac{k}{m}x_2$$

Pendulum solution and Lyapunov function



Solution of pendulum equation

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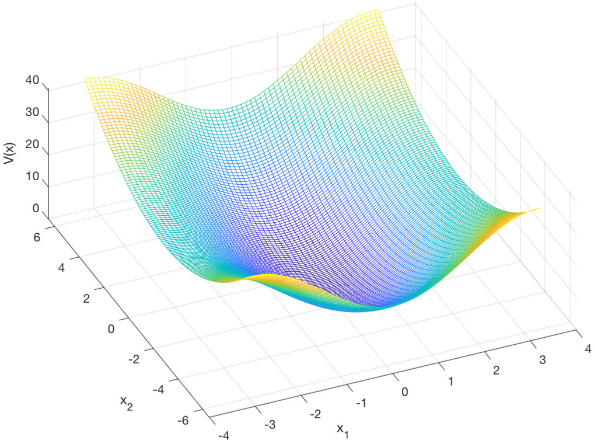

Lyapunov function $V(x) = x_2^2/2 + g(1 - \cos x_1) + 0.1 x_2 \sin(x_1)$

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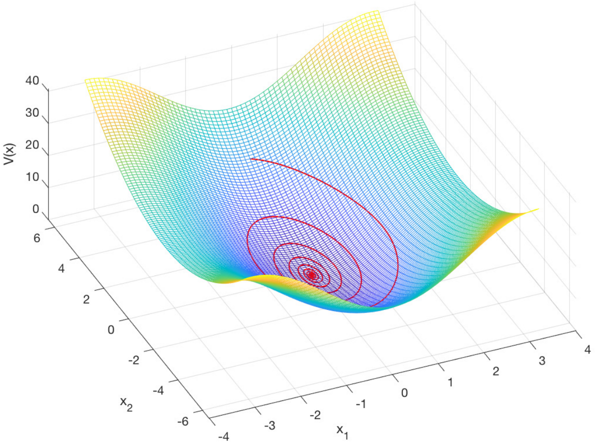

Lyapunov function with solution superimposed

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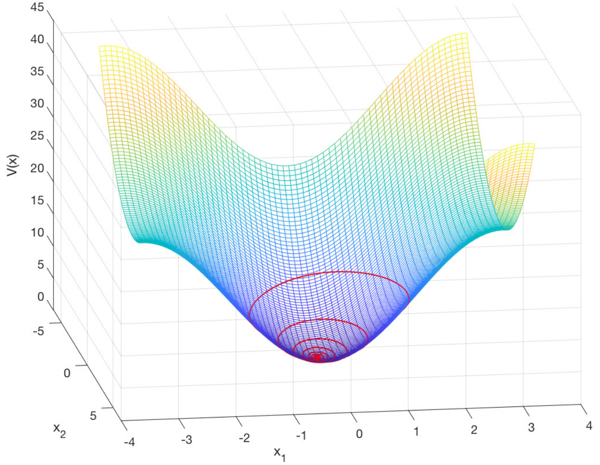

Lyapunov function with solution superimposed

Control Lyapunov functions

A smooth control Lyapunov function (clf) is characterised by

$$\alpha_1(\|x\|) \le V(x) \ \le \ \alpha_2(\|x\|)$$

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A clf hence acts like a road map, showing the way to the desired set or equilibrium

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Remedies:

- Replace $DV(x)f(x,u)$ in $(*)$ by the Dini derivative

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Remedies:

- Replace $DV(x)f(x,u)$ in $(*)$ by the Dini derivative
- Switch to discrete time and replace $(*)$ by

$$\inf_{u\in U} V(g(x,u)) \le V(x) - \alpha_3(\|x\|)$$

Numerical computation of Lyapunov functions

Various numerical approaches for computing (control) Lyapunov functions have been developed over the years:

- Series expansion [Kirin et al. '82]
- Semi-Lagrangian schemes [Camilli/Gr./Wirth '00, Falcone/Gr./Wirth '00]
- Finite elements and linear programming [Hafstein '02ff]
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Deep neural networks as approximation architecture are believed to mitigate this effort

Deep neural networks

Neural network approaches for Lyapunov functions

The computation of control Lyapunov functions via neural networks has been investigated since more than 30 years:

[Sontag '92, Long/Bayoumi '93, Abu-Khalaf/Lewis '04, Serpen '05, Petridis/Petridis '06, Noroozi/Karimaghaee/Safaei/Javadi '08, Khansari-Zadeh/Billard '14, Richards/Berkenkamp/Krause '18]

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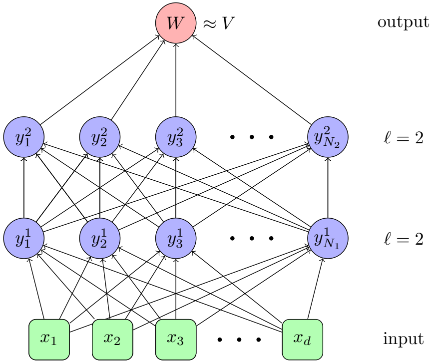
In what follows, I will present four recent developments in this field:

- Mitigating the curse of dimensionality
- Avoiding singularities
- Verification
- Nonsmooth clfs

Deep neural network with 2 hidden layers



$$W_\theta(x) = a \cdot y^2 + c$$

$$y^2_k = \sigma^2(w^2_k \cdot y^1 + b^2_k)$$

$$y^1_k = \sigma^1(w^1_k \cdot x + b^1_k)$$

w^1_k, w^2_k, a = vectors of weights, "$\cdot$" = scalar product

b^1_k, b^2_k, c = scalar parameters, $\sigma^1, \sigma^2 : \mathbb{R} \to \mathbb{R}$ = activation functions

Examples: $\sigma(r) = r$, $\sigma(r) = \max\{r, 0\}$, $\sigma(r) = \ln(e^r + 1)$, $\sigma(r) = \frac{1}{1+e^{-r}}$

θ = vector of all parameters $(w^\ell_k, b^\ell_k, a, c)$

$W_{\theta^*}(x) \approx V(x)$, approximated Lyapunov function for "trained" θ^*

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Training and minimizing over u

Training is usually done similar to learning PDE or ODE solutions with PINNs:

We minimize $\sum_i L(W\theta(x_i), D_x W_\theta(x_i), x_i)$ w.r.t. θ at sampling points x_i

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where the loss function L penalizes the violation of the inequalities that define the clf, e.g.,

$$L(w,p,x) = \left[\min_{u\in U} pf(x,u) + \alpha_3(\|x\|)\right]_+^2 + \nu\left([w-\alpha_1(\|x\|)]_-^2 + [w-\alpha_2(\|x\|)]_+^2\right)$$

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One possibility: If $f(x,u) = f_1(x) + f_2(x)u$ and $u \in [-C, C]^m$, then

$$\min_{u\in U} pf(x,u) = pf_1(x) - C\|pf_2(x)\|_1$$

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Mitigating the curse of dimensionality

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- For this reason, here we use separable functions

What are separable functions and why are they beneficial?

Separable function:

$$V(x) = \sum_{j=1}^{s} V_j(w_j), \quad w_j = \begin{pmatrix} x_{i_{j,1}} \\ \vdots \\ x_{i_{j,d_j}} \end{pmatrix}$$

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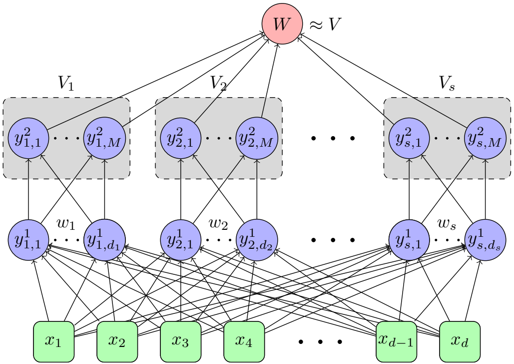
We approximate the individual V_j by the grey blocks, whose number grows linearly with the dimension d



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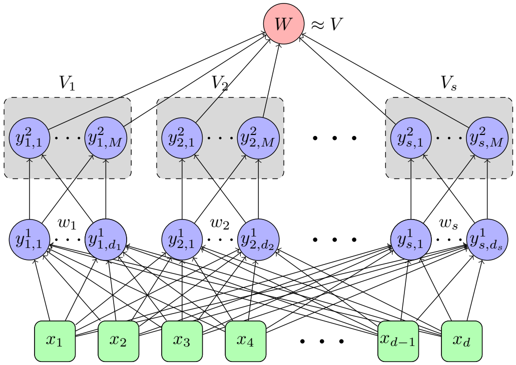

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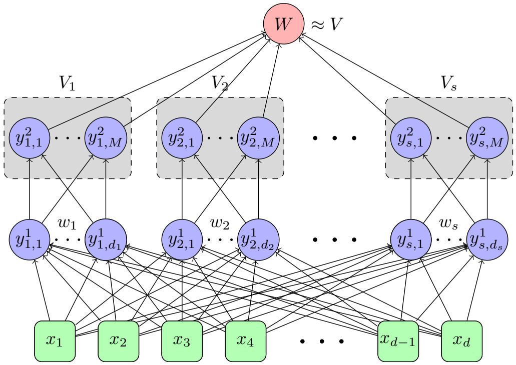

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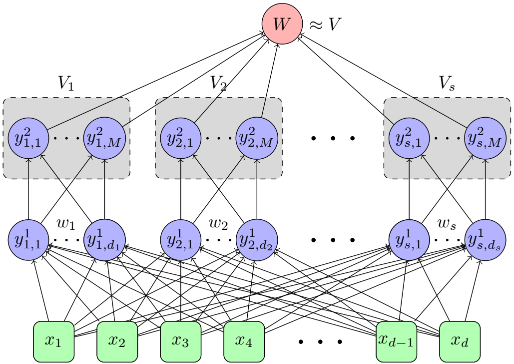
Applying the universal approximation theorem separately in each grey block, we can prove:

Theorem: Functions $V(x) = \sum_{j=1}^{s} V_j(w_j)$ and their derivatives, with $V_j \in \mathcal{W}_1^{d_j}$ and $d_j \leq d_{\max}$ independent of d can be approximated on K with any accuracy $\varepsilon > 0$ with a number of neurons growing only polynomially in d

What are separable functions and why are they beneficial?

Separable function:

$$V(x) = \sum_{j=1}^{s} V_j(w_j), \quad w_j = \begin{pmatrix} x_{i_{j,1}} \\ \vdots \\ x_{i_{j,d_j}} \end{pmatrix}$$



We approximate the individual V_j by the grey blocks, whose number grows linearly with the dimension d

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More precisely, the number of required neurons is $\mathcal{O}\left(\varepsilon^{-d_{\max}}\right) \mathcal{O}\left(d^{d_{\max}+1}\right)$

Nonlinear small-gain theory

For Lyapunov functions, nonlinear small gain theory provides existence results for separable Lyapunov functions

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It assumes that the system can be decomposed into subsystems

$$\dot{z}_i = f_i(z_i, z_{-i}), \quad i = 1, \ldots, s, \qquad z_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_s)$$

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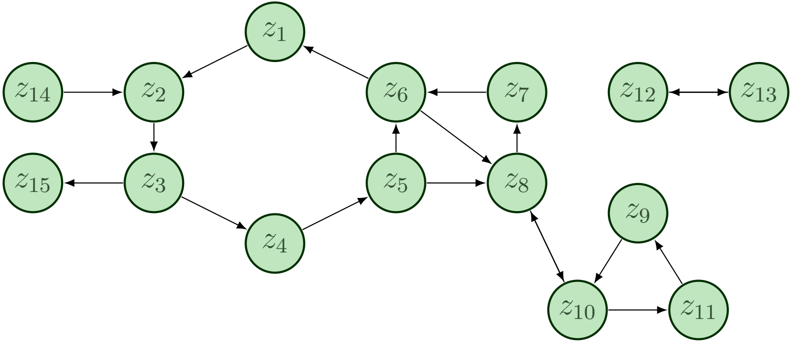

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then a separable Lyapunov function $V(x) = \sum_{j=1}^{s} V_j(z_j)$ exists

[Dashkovskiy/Rüffer/Wirth '10, Dashkovskiy/Ito/Wirth '11]

See also [Jiang/Teel/Praly '94, Jiang/Mareels/Wang '96, Rüffer '07ff, ...]

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Control Lyapunov functions

If we assume smoothness, a control Lyapunov function (clf) is characterised by

$$\alpha_1(\|x\|) \le V(x) \ \le \ \alpha_2(\|x\|)$$

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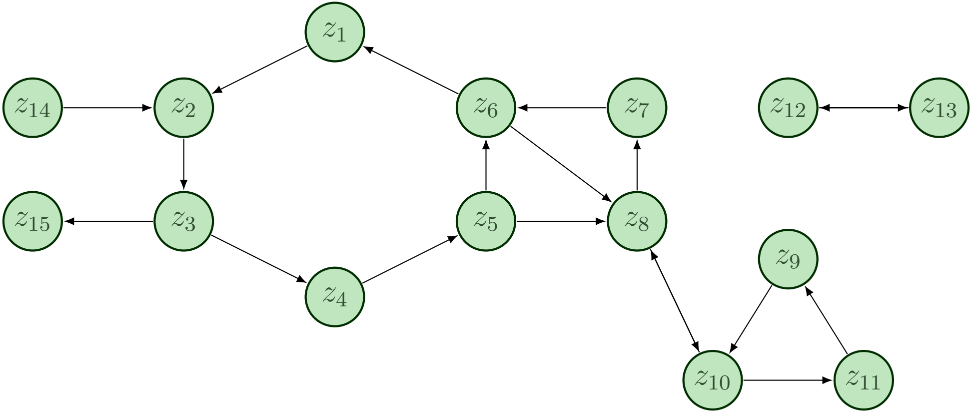
Rather, the network will "learn" this structure during the training process

Example for a suitable graph structure



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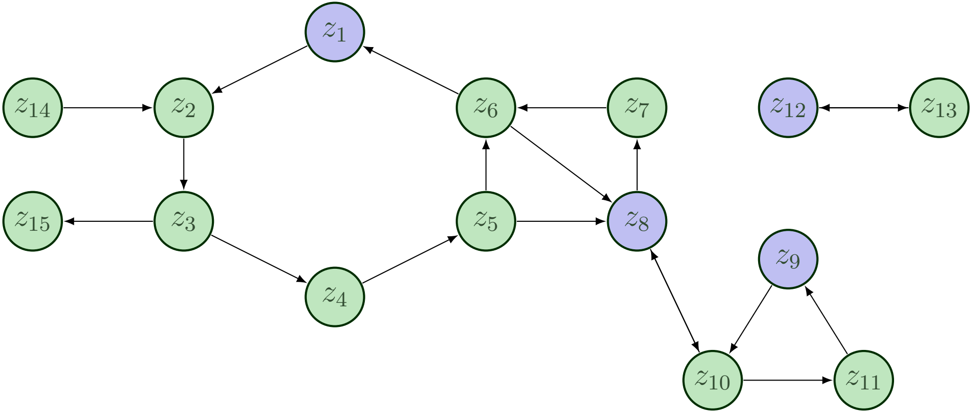

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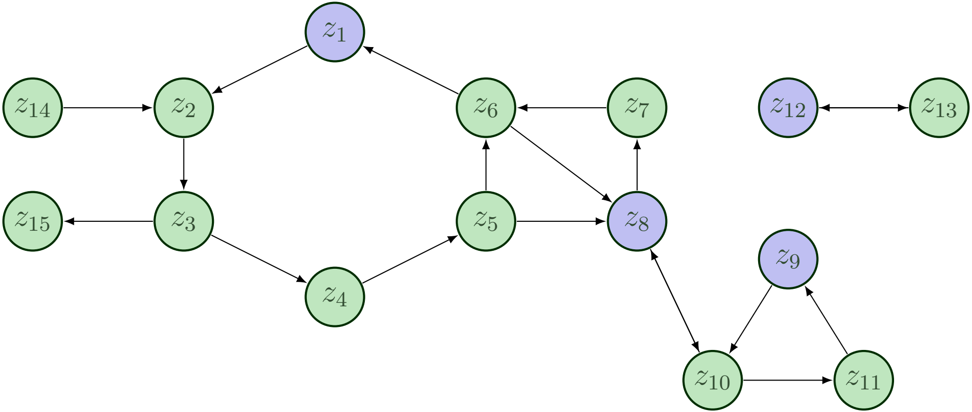

If the blue nodes are active $\rightsquigarrow$ $V(x) = \sum_{j=1}^{15} V_j(z_j)$ exists

Computation with DNN

Example:

$$
\begin{aligned}
\dot{x}_1 &= x_3 + u \\
\dot{x}_2 &= x_1 - x_2 + x_1^2 \\
\dot{x}_3 &= x_2 - x_3 \\
\dot{x}_4 &= x_3 - x_4 \\
\dot{x}_5 &= x_4 - x_5 \\
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\end{aligned}
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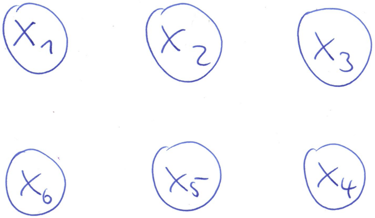

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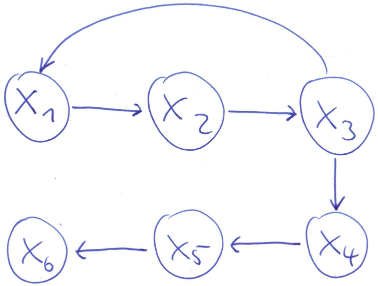

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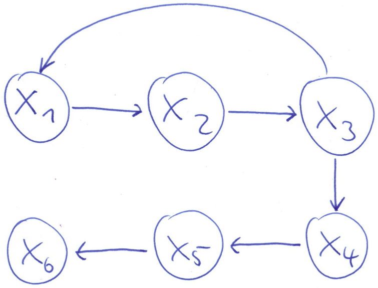

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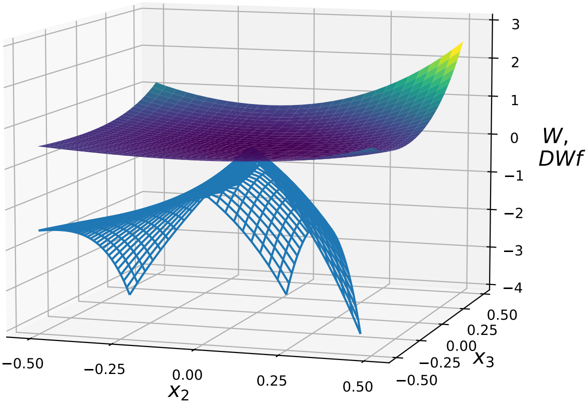
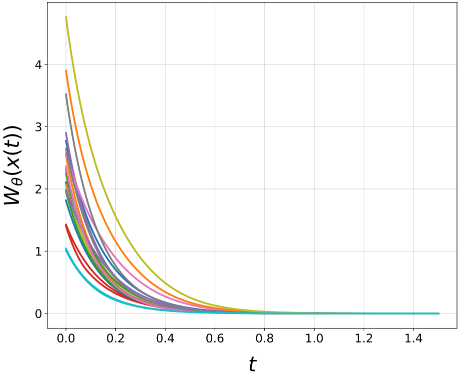


Computation time: 820s

Computation with DNN

Example [Ahmadi/Krstic/Parrilo '11]:

$$\dot{x} = \begin{pmatrix} -x_1 + x_1x_2 - 0.1x_9^2 \\ -x_2u_1 \\ -x_3 + x_3x_4 - 0.1x_1^2 \\ -x_4u_2 \\ -x_5 + x_5x_6 + 0.1x_7^2 \\ -x_6u_3 \\ -x_7 + x_7x_8 \\ -x_8u_4 \\ -x_9 + x_9x_{10} \\ -x_{10}u_5 + 0.1x_2^2 \end{pmatrix}$$



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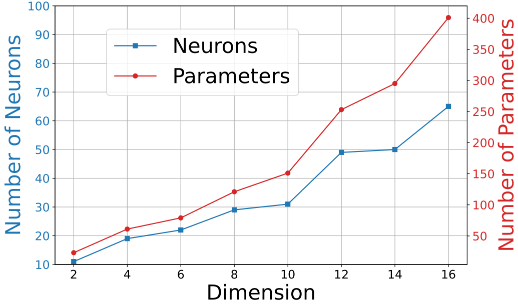

Avoiding singularities

The singularity problem

An approximate (control) Lyapunov function W_{θ^*} will only satisfy the inequalities

$$\alpha_1(\|x\|) \le W_{\theta^*}(x) \le \alpha_2(\|x\|)$$

$$\inf_{u\in U} DW_{\theta^*}(x)f(x,u) \le -\alpha_3(\|x\|)$$

approximately

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$\rightsquigarrow$ W_{θ^*} is only a Lyapunov function outside this region

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Idea [Barreau/Bastianello '25]: Represent the Lyapunov function as

$$V(x) \approx x^T P x + \sum_{i \in \mathcal{I}} W_{\theta,i}(x) \prod_{k=1}^{n} x_k^{i_k}$$

where $W_{\theta,i}(x)$, $i \in \mathcal{I}$ are the (scalar) output components of a neural network with high-dimensional output and

$$\mathcal{I} = \{i = (i_1, i_2, \dots, i_n) \in \{0,1,2,3\}^n \,|\, \sum_{k=1}^{n} i_k = 3\}$$

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Alternative approach: use more sampling points around 0 and include 0 in the sampling points

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While there are several papers in this direction, e.g. [Hafstein et al. 14ff; Liu et al. 24ff], using techniques from approximation theory or from formal verification, all of them seem to be prone to the curse of dimensionality

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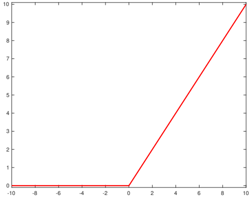

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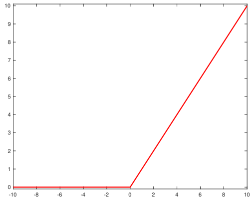

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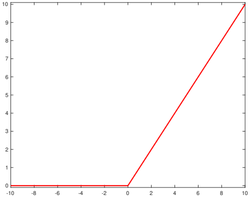

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Can we exploit this property?

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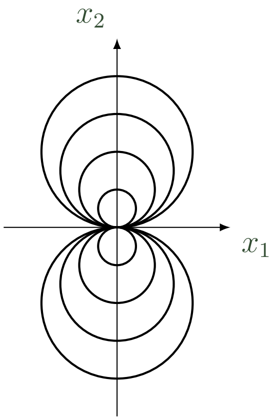

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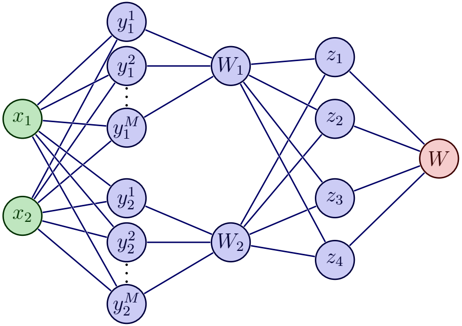

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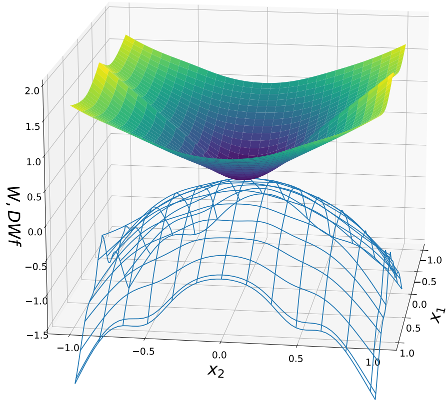

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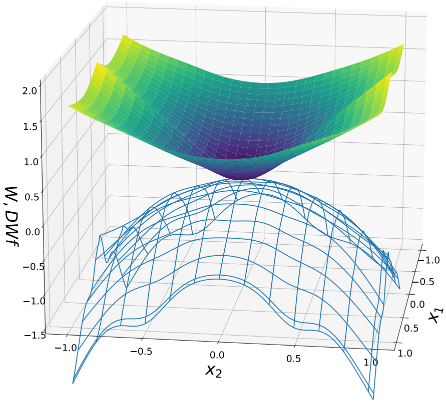

But: learning nonsmooth functions is a challenge

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  - for nonsmooth functions
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- extension to, e.g., control barrier functions
  - preliminary work of Jun Liu et al. and others exists

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